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Noncommutative Solitons and Quasideterminants

Masashi Hamanaka[1]

Nagoya University, Department of Mathematics,
Chikusa-ku, Nagoya, 464-8602, JAPAN

Abstract

We discuss extension of soliton theory and integrable systems to noncommutative spaces, focusing on integrable aspects of noncommutative anti-self-dual Yang-Mills equations. We give wide class of exact solutions by solving a Riemann-Hilbert problem for the Atiyah-Ward ansatz and present Bäcklund transformations for the $G = U(2)$ noncommutative anti-self-dual Yang-Mills equations. We find that one kind of noncommutative determinants, quasideterminants, play crucial roles in the construction of noncommutative solutions. We also discuss reduction of a noncommutative anti-self-dual Yang-Mills equation to noncommutative integrable equations. This is partially based on collaboration with C. R. Gilson and J. J. C. Nimmo (Glasgow).

[1] E-mail: hamamnaka@math.nagoya-u.ac.jp

1 Introduction

Extension of integrable systems and soliton theories to non-commutative (NC) space-times [2] has been studied by many authors for the last couple of years and various kinds of the integrable-like properties have been revealed. (For reviews, see [7, 31, 35, 47, 48, 59, 79].) This is partially motivated by recent developments of noncommutative gauge theories on D-branes. In the noncommutative gauge theories, noncommutative extension corresponds to presence of background flux and noncommutative solitons are, in some situations, just lower-dimensional D-branes themselves. Hence exact analysis of noncommutative solitons just leads to that of D-branes and various applications to D-brane dynamics have been successful [10, 30, 38, 76]. In this sense, noncommutative solitons plays important roles in noncommutative gauge theories.

Most of noncommutative integrable equations such as noncommutative KdV equations apparently belong not to gauge theories but to scalar theories. However now, it is proved that they can be derived from noncommutative anti-self-dual (ASD) Yang-Mills (YM) equations by reduction (e.g. [33, 32]), which is first conjectured explicitly by the author and K. Toda [36]. (Original commutative one is proposed by R. Ward [84] and hence this conjecture is sometimes called *noncommutative Ward's conjecture.*) Noncommutative anti-self-dual Yang-Mills equations belong to gauge theories and hence lower-dimensional many integrable equations which apparently belong to scalar theories must have physical situations (in the presence of background flux), and therefore analysis of exact soliton solutions of noncommutative integrable equations could be applied to the corresponding physical situations in the framework of N=2 string theory [42, 51, 56, 65].

Furthermore, integrable aspects of anti-self-dual Yang-Mills equation can be understood from a geometrical framework, the *twistor theory*. Via the Ward's conjecture, the twistor theory gives a new geometrical viewpoint into lower-dimensional integrable equations and some classification can be made in such a way. These results are summarized in the book of Mason and Woodhouse elegantly [58]. (See also [11, 12].)

In this article, we discuss integrable aspects of noncommutative anti-self-dual Yang-Mills equations from the viewpoint of noncommutative twistor theory. We give a series of noncommutative Atiyah-Ward ansatz solutions by solving a noncommutative Riemann-Hilbert problem. The solutions include not only noncommutative instantons (with finite action) but also noncommutative non-linear plane waves (with infinite action) and so on. We have also found that a kind of noncommutative determinants, the quasideterminants, play crucial roles in construction of exact solutions and present a direct proof of the results of the Bäcklund transformation and the generated solutions without twistor framework. These are due to collaboration with C. Gilson and J. Nimmo (Glasgow) [22, 23].

Finally we give an example of noncommutative Ward's conjecture, reduction of the noncommutative anti-self-dual Yang-Mills equation into the noncommutative KdV equation via the noncommutative toroidal KdV equation. The reduced equation actually has integrable-like properties such as infinite conserved quantities, exact N-soliton solutions and so on. These results would lead to a kind of classification of noncommutative inte-

[2] In the present article, the word "noncommutative" always refers to generalization to noncommutative spaces, not to non-abelian and so on.

grable equations from a geometrical viewpoint and to applications to the corresponding physical situations and geometry also.

2 Noncommutative anti-self-dual Yang-Mills equations

In this section, we review some aspects of noncommutative anti-self-dual Yang-Mills equation and establish notations.

2.1 Noncommutative gauge theories

Noncommutative spaces are defined by the noncommutativity of the coordinates:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{2.1}$$

where $\theta^{\mu\nu}$ are real constants called the *noncommutative parameters*. The noncommutative parameter is anti-symmetric with respect to μ,ν: $\theta^{\nu\mu} = -\theta^{\mu\nu}$ and the rank is even. This relation looks like the canonical commutation relation in quantum mechanics and leads to "space-space uncertainty relation." Hence the singularity which exists on commutative spaces could resolve on noncommutative spaces. This is one of the prominent features of noncommutative field theories and yields various new physical objects such as $U(1)$ instantons.

Noncommutative field theories are given by the exchange of ordinary products in the commutative field theories for the star-products and realized as deformed theories from the commutative ones. The ordering of non-linear terms are determined by some additional requirements such as gauge symmetry. The star-product is defined for ordinary fields on commutative spaces. For Euclidean spaces, it is explicitly given by

$$\begin{aligned} f \star g(x) &:= \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu^{(x')}\partial_\nu^{(x'')}\right) f(x')g(x'')\Big|_{x'=x''=x} \\ &= f(x)g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x) + O(\theta^2), \end{aligned} \tag{2.2}$$

where $\partial_\mu^{(x')} := \partial/\partial x'^\mu$ and so on. This explicit representation is known as the *Moyal product* [60]. The star-product has associativity: $f \star (g \star h) = (f \star g) \star h$, and returns back to the ordinary product in the commutative limit: $\theta^{\mu\nu} \to 0$. The modification of the product makes the ordinary spatial coordinate "noncommutative," that is, $[x^\mu, x^\nu]_\star := x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$.

We note that the fields themselves take c-number values as usual and the differentiation and the integration for them are well-defined as usual, for example, $\partial_\mu \star \partial_\nu = \partial_\mu\partial_\nu$, and the wedge product of $\lambda = \lambda_\mu(x)dx^\mu$ and $\rho = \rho_\nu(x)dx^\nu$is $\lambda_\mu \star \rho_\nu dx^\mu \wedge dx^\nu$.

Noncommutative gauge theories are defined in this way by imposing noncommutative version of gauge symmetry, where the gauge transformation is defined as follows:

$$A_\mu \to g^{-1} \star A_\mu \star g + g^{-1} \star \partial_\mu g, \tag{2.3}$$

where g is an element of the gauge group G (The inverse is always supposed to mean in the sense of the star product in this article.) This is sometimes called the *star gauge transformation*. We note that because of noncommutativity, the commutator terms in field strength are always needed even when the gauge group is abelian in order to preserve the star gauge symmetry. This $U(1)$ part of the gauge group actually plays crucial roles in general. We note that because of noncommutativity of matrix elements, cyclic symmetry of traces is broken in general:

$$\text{Tr } A \star B \neq \text{Tr } B \star A. \tag{2.4}$$

Therefore, gauge invariant quantities becomes hard to define on noncommutative spaces.

2.2 Noncommutative anti-self-dual Yang-Mills equations

Let us consider Yang-Mills theories in 4-dimensional noncommutative spaces whose real coordinates of the space are denoted by (x^0, x^1, x^2, x^3), where the gauge group is $GL(N, \mathbb{C})$. Here, we follow the convention in [58].

First, we introduce double null coordinates of 4-dimensional space as follows

$$ds^2 = 2(dzd\tilde{z} - dwd\tilde{w}), \tag{2.5}$$

We can recover various kind of real spaces by putting the corresponding reality conditions on the double null coordinates $z, \tilde{z}, w, \tilde{w}$ as follows:

- Euclidean Space ($\bar{w} = -\tilde{w}; \bar{z} = \tilde{z}$): An example is

$$\begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^1 & -(x^2 - ix^3) \\ x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}. \tag{2.6}$$

- Minkowski Space ($\bar{w} = \tilde{w}$; z and $\tilde{z}$ are real.): An example is

$$\begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - x^1 \end{pmatrix}. \tag{2.7}$$

- Ultrahyperbolic Space ($\bar{w} = \tilde{w}; \bar{z} = \tilde{z}$): Example are

$$\begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}, \quad \text{or} \quad z, \tilde{z}, w, \tilde{w} \in \mathbb{R}. \tag{2.8}$$

The coordinate vectors $\partial_z, \partial_z.\partial_{\tilde{w}}, \partial_{\tilde{z}}$ form a null tetrad and are represented explicitly as:

$$\begin{aligned} \partial_z &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^0} + i\frac{\partial}{\partial x^1}\right), \quad \partial_{\tilde{z}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^0} - i\frac{\partial}{\partial x^1}\right), \\ \partial_w &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^2} + i\frac{\partial}{\partial x^3}\right), \quad \partial_{\tilde{w}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^2} - i\frac{\partial}{\partial x^3}\right). \end{aligned} \tag{2.9}$$

For Euclidean and ultrahyperbolic signatures, Hodge dual operator $*$ satisfies $*^2 = 1$ and hence the space of 2-forms β decomposes into the direct sum of eigenvalues of $*$ with eigenvalues ± 1, that is, self-dual (SD) part ($*\beta = \beta$) and anti-self-dual (ASD) part ($*\beta = -\beta$). From now on, we treat these two signatures.

Typical examples of self-dual forms are

$$\alpha = dw \wedge dz, \quad \tilde{\alpha} = d\tilde{w} \wedge d\tilde{z}, \quad \omega = dw \wedge d\tilde{w} - dz \wedge d\tilde{z}, \tag{2.10}$$

and those of anti-self-dual forms are

$$dw \wedge d\tilde{z}, \quad d\tilde{w} \wedge dz, \quad dw \wedge d\tilde{w} + dz \wedge d\tilde{z}. \tag{2.11}$$

Noncommutative anti-self-dual Yang-Mills equation is derived from compatibility condition of the following linear system:

$$\begin{aligned} L \star \psi &:= (D_w - \zeta D_{\tilde{z}}) \star \psi = (\partial_w + A_w - \zeta(\partial_{\tilde{z}} + A_{\tilde{z}})) \star \psi(x;\zeta) = 0, \\ M \star \psi &:= (D_z - \zeta D_{\tilde{w}}) \star \psi = (\partial_z + A_z - \zeta(\partial_{\tilde{w}} + A_{\tilde{w}})) \star \psi(x;\zeta) = 0, \end{aligned} \tag{2.12}$$

where $A_z, A_w, A_{\tilde{z}}, A_{\tilde{w}}$ and $D_z, D_w, D_{\tilde{z}}, D_{\tilde{w}}$ denote gauge fields and covariant derivatives in the Yang-Mills theory, respectively. The constant ζ is called the *spectral parameter*. We note that $h(x) = \psi(x, \zeta = 0), \tilde{h}(x) = \tilde{\psi}(x, \zeta = \infty)$.

The compatible condition $[L, M]_\star = 0$, gives rise to a quadratic polynomial of ζ and each coefficient yields the following equations:

$$\begin{aligned} F_{wz} &= \partial_w A_z - \partial_z A_w + [A_w, A_z]_\star = 0, \\ F_{\tilde{w}\tilde{z}} &= \partial_{\tilde{w}} A_{\tilde{z}} - \partial_{\tilde{z}} A_{\tilde{w}} + [A_{\tilde{w}}, A_{\tilde{z}}]_\star = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} &= \partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z + \partial_{\tilde{w}} A_w - \partial_w A_{\tilde{w}} + [A_z, A_{\tilde{z}}]_\star - [A_w, A_{\tilde{w}}]_\star = 0, \end{aligned} \tag{2.13}$$

which are equivalent to the noncommutative anti-self-dual Yang-Mills equations $F_{\mu\nu} = - * F_{\mu\nu}$ in the real representation.

Gauge transformations act on the linear system (2.12) as

$$L \mapsto g^{-1} \star L \star g, \ M \mapsto g^{-1} \star M \star g, \ \psi \mapsto g^{-1} \star \psi, \quad g \in G. \tag{2.14}$$

We note that the solution ψ ($N \times N$ matrix) of the linear system (2.12) is not regular at $\zeta = \infty$ because of Liouville's theorem. (If it is regular, then the gauge fields become flat.) Hence we have to consider another linear system on another local patch whose coordinate is $\tilde{\zeta} = 1/\zeta$ as

$$\begin{aligned} \tilde{L} \star \tilde{\psi} &:= \tilde{\zeta} D_w \star \tilde{\psi} - D_{\tilde{z}} \star \tilde{\psi} = 0, \\ \tilde{M} \star \tilde{\psi} &:= \tilde{\zeta} D_z \star \tilde{\psi} - D_{\tilde{w}} \star \tilde{\psi} = 0. \end{aligned} \tag{2.15}$$

The compatibility condition of this another linear systems also gives rise to the anti-self-dual Yang-Mills equation.

2.3 Noncommutative Yang's equations and J, K-matrices

Here we discuss the potential forms of noncommutative anti-self-dual Yang-Mills equations such as noncommutative J, K-matrix formalisms and noncommutative Yang's equation, which is already presented by e.g. K. Takasaki [78].

Let us first discuss the *J-matrix formalism* of noncommutative anti-self-dual Yang-Mills equation. The first equation of noncommutative anti-self-dual Yang-Mills equation (2.13) is the compatible condition of $D_z \star h = 0, D_w \star h = 0$, where h is a $N \times N$ matrix. Hence we get

$$A_z = -(\partial_z h) \star h^{-1}, \quad A_w = -(\partial_w h) \star h^{-1}. \tag{2.16}$$

Similarly, the second equation of noncommutative anti-self-dual Yang-Mills equation (2.13) leads to

$$A_{\tilde{z}} = -(\partial_{\tilde{z}} \tilde{h}) \star \tilde{h}^{-1}, \quad A_{\tilde{w}} = -(\partial_{\tilde{w}} \tilde{h}) \star \tilde{h}^{-1}, \tag{2.17}$$

where $\tilde{h}$ is also a $N \times N$ matrix. By defining $J = \tilde{h}^{-1} \star h$, the third equation of noncommutative anti-self-dual Yang-Mills equation (2.13) becomes noncommutative Yang's equation

$$\partial_z (J^{-1} \star \partial_{\tilde{z}} J) - \partial_w (J^{-1} \star \partial_{\tilde{w}} J) = 0, \tag{2.18}$$

or equivalently,

$$\partial \left(J^{-1} \star \tilde{\partial} J \right) \wedge \omega = 0. \tag{2.19}$$

where $\partial = dw\partial_w + dz\partial_z,\ \tilde{\partial} = d\tilde{w}\partial_{\tilde{w}} + d\tilde{z}\partial_{\tilde{z}}$ ω is the same one as in (2.10).

Gauge transformations act on h and $\tilde{h}$ as

$$h \mapsto g^{-1} h,\ \tilde{h} \mapsto g^{-1} \tilde{h}, \quad g \in G. \tag{2.20}$$

Hence the Yang's J-matrix is gauge invariant while the matrices h and $\tilde{h}$ are gauge dependent. In this article, we sometimes use the following gauge for $G = GL(2)$:

$$\begin{aligned} h_{\mathrm{MW}} &= \begin{pmatrix} f & 0 \\ e & 1 \end{pmatrix}, \quad \tilde{h}_{\mathrm{MW}} = \begin{pmatrix} 1 & g \\ 0 & b \end{pmatrix}, \\ \text{then} \quad J &= \tilde{h}_{\mathrm{MW}}^{-1} \star h_{\mathrm{MW}} = \begin{pmatrix} f - g \star b^{-1} \star e & -g \star b^{-1} \\ b^{-1} \star e & b^{-1} \end{pmatrix}. \end{aligned} \tag{2.21}$$

which is called the *Mason-Woodhouse gauge*.

There is another potential form of noncommutative anti-self-dual Yang-Mills equation, known as the *K-matrix formalism*. In the gauge in which $A_w = A_z = 0$, the third equation of (2.13) becomes $\partial_z A_{\tilde{z}} - \partial_w A_{\tilde{w}} = 0$. This implies the existence of a potential K such that $A_{\tilde{z}} = \partial_w K, A_{\tilde{w}} = \partial_z K$. Then the second equation of (2.13) becomes

$$\partial_z \partial_{\tilde{z}} K - \partial_w \partial_{\tilde{w}} K + [\partial_w K, \partial_z K]_\star = 0. \tag{2.22}$$

Under the gauge $A_w = A_z = 0$, we get

$$\psi = 1 + \zeta K + \mathcal{O}(\zeta^2), \quad \tilde{\psi} = J^{-1} + \mathcal{O}(\tilde{\zeta}), \tag{2.23}$$

and $A_{\tilde{w}} = J^{-1} \star \partial_{\tilde{w}} J = \partial_z K$, $A_{\tilde{z}} = J^{-1} \star \partial_{\tilde{z}} J = \partial_w K$. This gauge is suitable for the discussion of (binary) Darboux transformations for (noncommutative) anti-self-dual Yang-Mills equation [24, 64, 70].

3 Twistor description of noncommutative anti-self-dual Yang-Mills equations

In this section, we construct wide class of exact solutions of noncommutative anti-self-dual Yang-Mills equations from the geometrical viewpoint of noncommutative twistor theory. Noncommutative twistor theory has been developed by several authors and the mathematical foundation is established [2, 37, 45, 78].

Twistor theory is roughly said to be based on a correspondence between (complexified) space-time coordinates $(z, \tilde{z}, w, \tilde{w})$ and twistor coordinates (λ, μ, ζ) which are local coordinates of a 3-dimensional complex projective space (twistor space). There is a relation between them as follows:

$$\lambda = \zeta w + \tilde{z}, \;\; \mu = \zeta z + \tilde{w}, \tag{3.1}$$

which is called the *incidence relation.* This relation implies that for any twistor function $f(\lambda, \mu, \zeta)$,

$$\begin{aligned} l f(\lambda, \mu, \zeta) &:= (\partial_w - \zeta \partial_{\tilde{z}}) f(\lambda, \mu, \zeta) = 0, \\ m f(\lambda, \mu, \zeta) &:= (\partial_z - \zeta \partial_{\tilde{w}}) f(\lambda, \mu, \zeta) = 0. \end{aligned} \tag{3.2}$$

3.1 Noncommutative Penrose-Ward transformation

For anti-self-dual Yang-Mills theory, there is a one-to-one correspondence between solutions of the anti-self-dual Yang-Mills equation and holomorphic vector bundles on the twistor space. The former is given by solutions $\psi, \tilde{\psi}$ of the linear systems (2.12) and (2.15). The latter is given by patching matrices P of the holomorphic vector bundles. The explicit correspondence is called the *Penrose-Ward correspondence.*

Here we just need the Moyal-deformed Penrose-Ward correspondence between anti-self-dual Yang-Mills solutions $\psi, \tilde{\psi}$ and the patching matrix P.

From given ψ and $\tilde{\psi}$, the patching matrix P is constructed as

$$P(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) = \tilde{\psi}^{-1}(x; \zeta) \star \psi(x; \zeta). \tag{3.3}$$

(Here we note that $\psi(x; \zeta)$ is regular w.r.t. ζ around $\zeta = 0$ and $\tilde{\psi}(x; \zeta)$ is regular w.r.t. $\tilde{\zeta}$ around $\tilde{\zeta} = 0$ or equivalently $\zeta = \infty$.) Conversely, if for a given P, there exists the factorization (**??**) into ψ and $\tilde{\psi}$ where $\psi(x; \zeta)$ is regular w.r.t. ζ around $\zeta = 0$

and $\tilde{\psi}(x;\zeta)$ is regular w.r.t. $\tilde{\zeta}$ around $\tilde{\zeta}=0$, then the ψ and $\tilde{\psi}$ are solutions of linear systems (2.12) and (2.15) for noncommutative anti-self-dual Yang-Mills equations. (This factorization problem is called the *Riemann-Hilbert problem* and solved formally [78]. Noncommutativity can be introduced into only two variables $\zeta w+\tilde{z}$ and $\zeta z+\tilde{w}$. Then ζ is a commutative variable and steps of solving Riemann-Hilbert problem become similar to commutative one.)

3.2 Noncommutative Atiyah-Ward ansatz solutions for $G=GL(2)$

From now on, we restrict ourselves to $G=GL(2)$. In this case, we can take a simple ansatz for the Patching matrix P, which is called the Atiyah-Ward ansatz in commutative case [1]. Noncommutative generalization of this ansatz is straightforward and actually leads to a solution of the factorization problem. The l-th order noncommutative Atiyah-Ward ansatz is specified by the following form of the patching matrix up to constant matrix action from both sides ($l=0,1,2,\cdots$):

$$P_l(x;\zeta)=\left(\begin{array}{cc} 0 & \zeta^{-l} \\ \zeta^{l} & \Delta(x;\zeta)\end{array}\right). \tag{3.4}$$

We note that P satisfies eq. (3.2) and hence, the Laurent expansion of Δ w.r.t. ζ

$$\Delta(x;\zeta)=\sum_{i=-\infty}^{\infty}\Delta_i(x)\zeta^{-i}, \tag{3.5}$$

gives rise to the following recurrence relations in the coefficients as

$$\frac{\partial\Delta_i}{\partial z}=\frac{\partial\Delta_{i+1}}{\partial\tilde{w}},\quad \frac{\partial\Delta_i}{\partial w}=\frac{\partial\Delta_{i+1}}{\partial\tilde{z}}. \tag{3.6}$$

The wave functions ψ and $\tilde{\psi}$ can be expanded by ζ and $\tilde{\zeta}=1/\zeta$, respectively:

$$\begin{aligned}\psi &= h+\mathcal{O}(\zeta)=\left(\begin{array}{cc} h_{11}+\sum_{i=1}^{\infty}a_i\zeta^i & h_{12}+\sum_{i=1}^{\infty}b_i\zeta^i \\ h_{21}+\sum_{i=1}^{\infty}c_i\zeta^i & h_{22}+\sum_{i=1}^{\infty}d_i\zeta^i.\end{array}\right),\\ \tilde{\psi} &= \tilde{h}+\mathcal{O}(\tilde{\zeta})=\left(\begin{array}{cc} \tilde{h}_{11}+\sum_{i=1}^{\infty}\tilde{a}_i\tilde{\zeta}^i & \tilde{h}_{12}+\sum_{i=1}^{\infty}\tilde{b}_i\tilde{\zeta}^i \\ \tilde{h}_{21}+\sum_{i=1}^{\infty}\tilde{c}_i\tilde{\zeta}^i & \tilde{h}_{22}+\sum_{i=1}^{\infty}\tilde{d}_i\tilde{\zeta}^i.\end{array}\right).\end{aligned} \tag{3.7}$$

Now let us solve the factorization problem $\tilde{\psi}\star P=\psi$ for the noncommutative Atiyah-Ward ansatz. This is concretely written down as

$$\left(\begin{array}{cc}\tilde{\psi}_{11} & \tilde{\psi}_{12}\\ \tilde{\psi}_{21} & \tilde{\psi}_{22}\end{array}\right)\star\left(\begin{array}{cc} 0 & \zeta^{-l} \\ \zeta^{l} & \Delta(x;\zeta)\end{array}\right)=\left(\begin{array}{cc}\psi_{11} & \psi_{12}\\ \psi_{21} & \psi_{22}\end{array}\right), \tag{3.8}$$

that is,

$$\tilde{\psi}_{12}\zeta^l=\psi_{11},\quad \tilde{\psi}_{22}\zeta^l=\psi_{21}, \tag{3.9}$$
$$\tilde{\psi}_{11}\zeta^{-l}+\tilde{\psi}_{12}\star\Delta=\psi_{12},\quad \tilde{\psi}_{21}\zeta^{-l}+\tilde{\psi}_{22}\star\Delta=\psi_{22}. \tag{3.10}$$

From Eqs. (3.7) and (3.9) we find that some entries become polynomials w.r.t. ζ:

$$\begin{aligned}
\psi_{11} &= h_{11}+a_1\zeta+a_2\zeta^2+\cdots a_{l-1}\zeta^{l-1}+\tilde{h}_{12}\zeta^l,\\
\psi_{21} &= h_{21}+b_1\zeta+b_2\zeta^2+\cdots b_{l-1}\zeta^{l-1}+\tilde{h}_{22}\zeta^l,\\
\tilde{\psi}_{12} &= \tilde{h}_{12}+a_{l-1}\zeta^{-1}+a_{l-2}\zeta^{-2}+\cdots+a_1\zeta^{1-l}+h_{11}\zeta^{-l},\\
\tilde{\psi}_{22} &= \tilde{h}_{22}+b_{l-1}\zeta^{-1}+b_{l-2}\zeta^{-2}+\cdots+b_1\zeta^{1-l}+h_{21}\zeta^{-l},
\end{aligned}\tag{3.11}$$

and so on. By substituting these relations into eq. (3.10), we get sets of equations for h and $\tilde{h}$ in the coefficients of $\zeta^0,\ \zeta^{-1},\ \cdots,\ \zeta^{-l}$:

$$\begin{aligned}
(h_{11},a_1,\cdots,a_{l-1},\tilde{h}_{12})D_{l+1}=(-\tilde{h}_{11},0,\cdots,0,h_{12}),\\
(h_{21},c_1,\cdots,c_{l-1},\tilde{h}_{22})D_{l+1}=(-\tilde{h}_{21},0,\cdots,0,h_{22}),
\end{aligned}\tag{3.12}$$

where

$$D_l:=\begin{pmatrix}\Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l}\\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l}\\ \vdots & \vdots & \ddots & \vdots\\ \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0\end{pmatrix}.\tag{3.13}$$

These linear equations can be solved by taking inverse matrix of D_{l+1} from right side and can be rewritten in terms of quasideterminants:

$$\begin{aligned}
h_{11} &= h_{12}\star|D_{l+1}|_{1,l+1}^{-1}-\tilde{h}_{11}\star|D_{l+1}|_{1,1}^{-1},\\
h_{21} &= h_{22}\star|D_{l+1}|_{1,l+1}^{-1}-\tilde{h}_{21}\star|D_{l+1}|_{1,1}^{-1},\\
\tilde{h}_{12} &= h_{12}\star|D_{l+1}|_{l+1,l+1}^{-1}-\tilde{h}_{1,1}\star|D_{l+1}|_{l+1,1}^{-1},\\
\tilde{h}_{22} &= h_{22}\star|D_{l+1}|_{l+1,l+1}^{-1}-\tilde{h}_{21}\star|D_{l+1}|_{l+1,1}^{-1}.
\end{aligned}\tag{3.14}$$

Here if we taking the Mason-Woodhouse gauge (2.22), Eq. (3.14) can be solved for h and $\tilde{h}$ in terms of quasideterminants of D_{l+1}:

$$\begin{aligned}
f=h_{11} &= -\begin{vmatrix}\boxed{\Delta_0} & \Delta_{-1} & \cdots & \Delta_{-l}\\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l}\\ \vdots & \vdots & \ddots & \vdots\\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0\end{vmatrix}^{-1}, \quad e=h_{21}=\begin{vmatrix}\Delta_0 & \Delta_{-1} & \cdots & \boxed{\Delta_{-l}}\\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l}\\ \vdots & \vdots & \ddots & \vdots\\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0\end{vmatrix}^{-1},\\
g=\tilde{h}_{12} &= -\begin{vmatrix}\Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l}\\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l}\\ \vdots & \vdots & \ddots & \vdots\\ \boxed{\Delta_l} & \Delta_{l-2} & \cdots & \Delta_0\end{vmatrix}^{-1}, \quad b=\tilde{h}_{22}=\begin{vmatrix}\Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l}\\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l}\\ \vdots & \vdots & \ddots & \vdots\\ \Delta_l & \Delta_{l-1} & \cdots & \boxed{\Delta_0}\end{vmatrix}^{-1}
\end{aligned}\tag{3.15}$$

This is the l-th order noncommutative Atiyah-Ward ansatz solution. For $l=0$, the noncommutative anti-self-dual Yang-Mills equation becomes a noncommutative linear equation $(\partial_z\partial_{\tilde{z}}-\partial_w\partial_{\tilde{w}})\Delta_0=0$. (We note that for the Euclidean space, this is the noncommutative Laplace equation because of the reality condition $\bar{w}=-\tilde{w}$. The fundamental solutions leads to instanton solutions.) Other scalar functions $\Delta_i(x)$ is determined explicitly by the recurrence relation (3.6) from the solution $\Delta_0(x)$ of this linear equation up to integral constants. Hence the noncommutative Atiyah-Ward ansatz solutions are exact.

3.3 Bäcklund transformation for the noncommutative Atiyah-Ward ansatz solutions

Finally let us discuss an adjoint action for the patching matrices $\alpha : P_l \mapsto P_{l+1} = A^{-1}P_l A$ in twistor side. This leads to a Bäcklund transformation for the noncommutative anti-self-dual Yang-Mills equation in Yang-Mills side, which is a noncommutative generalization of Corrigan-Fairlie-Yates-Goddard (CFYG) transformation [4].

The adjoint action is easily found and can be described by the following two kinds of adjoint actions:

$$\alpha = \beta \circ \gamma_0, \quad \beta : P \mapsto P^{\rm new} = B^{-1}PB, \quad \gamma_0 : P \mapsto P^{\rm new} = C_0^{-1}PC_0, \tag{3.16}$$

where

$$A = BC, \quad B = \begin{pmatrix} 0 & 1 \\ \zeta^{-1} & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.17}$$

In order to find the corresponding transformations in the Yang-Mills side, we have to observe how the adjoint actions act on the matrices h and $\tilde{h}$, that is, ψ and $\tilde{\psi}$. Here we take the Mason-Woodhouse gauge (2.22) again.

The γ_0-transformation is $h \mapsto hC_0, \tilde{h} \mapsto \tilde{h}C_0$ and hence $J = \tilde{h}^{-1} \star h \mapsto C_0^{-1}JC_0$. In the Mason-Woodhouse gauge, we can read explicit form of the transformations for the variables b, e, f, g in (2.22).

The β-transformation for ψ and $\tilde{\psi}$ must be defined together with a singular gauge transformation due to the Birkhoff factorization as follows:

$$\psi^{\rm new} = s \star \psi\ B, \quad \tilde{\psi}^{\rm new} = s \star \tilde{\psi}\ B, \tag{3.18}$$

where

$$s = \begin{pmatrix} 0 & \zeta b^{-1} \\ -f^{-1} & 0 \end{pmatrix}. \tag{3.19}$$

The explicit calculation gives

$$\psi^{\rm new} = \begin{pmatrix} b^{-1}\psi_{22} & -\zeta b^{-1} \star \psi_{21} \\ -\zeta^{-1} f^{-1} \star \psi_{12} & f^{-1} \star \psi_{11} \end{pmatrix}, \tag{3.20}$$

where ψ_{ij} is the (i,j)-th element of ψ. In the $\zeta \to 0$ limit, this reduces in the Mason-Woodhouse gauge to

$$h^{\rm new} = \begin{pmatrix} f^{\rm new} & 0 \\ e^{\rm new} & 1 \end{pmatrix} = \begin{pmatrix} b^{-1} & 0 \\ -f^{-1} \star j_{12} & 1 \end{pmatrix}, \tag{3.21}$$

where $\psi = h + j\zeta + \mathcal{O}(\zeta^2)$.

Here we note that the linear systems can be represented in terms of b, f, e, g as

$$\begin{aligned} L \star \psi &= (\partial_w - \zeta \partial_{\tilde{z}}) \star \psi + \begin{pmatrix} -f_w \star f^{-1} & \zeta g_{\tilde{z}} \star b^{-1} \\ -e_w \star f^{-1} & \zeta b_{\tilde{z}} \star b^{-1} \end{pmatrix} \star \psi = 0, \\ M \star \psi &= (\partial_z - \zeta \partial_{\tilde{w}}) \star \psi + \begin{pmatrix} -f_z \star f^{-1} & \zeta g_{\tilde{w}} \star b^{-1} \\ -e_z \star f^{-1} & \zeta b_{\tilde{w}} \star b^{-1} \end{pmatrix} \star \psi = 0. \end{aligned} \tag{3.22}$$

By picking the first order term of ζ in the 1-2 component of the first equation, we get

$$\partial_w(f^{-1} \star j_{12}) = -f^{-1} \star g_{\tilde{z}} \star b^{-1}. \tag{3.23}$$

Hence from the 1-1 and 2-1 components of Eq. (3.21), we have

$$f^{\mathrm{new}} = b^{-1}, \quad \partial_w e^{\mathrm{new}} = \partial_w(f^{-1} \star j_{12}) = -f^{-1} \star g_{\tilde{z}} \star b^{-1}. \tag{3.24}$$

In similar way, we can get the other ones.

3.4 Summary and comments

Here we can reconsider that the noncommutative Atiyah-Ward ansatz solutions are generated by the two kind of Bäcklund transformation from the seed solutions $b = e = f = g = \Delta^{-1}$ without solving the Riemann-Hilbert problem. (The difference of signs in f, g is not essential because they can be absorbed into the reflection symmetry $f \mapsto -f, g \mapsto -g$ of the noncommutative Yang equation.) From this viewpoint, let us summarize the previous results.

- β-transformation [58, 33]:

$$\begin{aligned}
&e_w^{\mathrm{new}} = -f^{-1} \star g_{\tilde{z}} \star b^{-1}, e_z^{\mathrm{new}} = -f^{-1} \star g_{\tilde{w}} \star b^{-1},\\
&g_{\tilde{z}}^{\mathrm{new}} = -b^{-1} \star e_w \star f^{-1}, g_{\tilde{w}}^{\mathrm{new}} = -b^{-1} \star e_z \star f^{-1},\\
&f^{\mathrm{new}} = b^{-1}, b^{\mathrm{new}} = f^{-1}.
\end{aligned} \tag{3.25}$$

- γ_0-transformation [22]:

$$\begin{pmatrix} f^{\mathrm{new}} & g^{\mathrm{new}} \\ e^{\mathrm{new}} & b^{\mathrm{new}} \end{pmatrix} = \begin{pmatrix} b & e \\ g & f \end{pmatrix}^{-1} = \begin{pmatrix} (b - e \star f^{-1} \star g)^{-1} & (g - f \star e^{-1} \star b)^{-1} \\ (e - b \star g^{-1} \star f)^{-1} & (f - g \star b^{-1} \star e)^{-1} \end{pmatrix} \tag{3.26}$$

We note that both transformations are *involutive*, that is, $\beta \circ \beta$ and $\gamma_0 \circ \gamma_0$ are the identity transformations.

$$\begin{array}{ccccccccccccc}
R_0 & \overset{\alpha}{\to} & & R_1 & & \overset{\alpha}{\to} & & R_2 & & \overset{\alpha}{\to} & & R_3 & \to & \cdots \\
& & \beta \searrow & & \gamma_0 \updownarrow & & \beta \searrow & & \gamma_0 \updownarrow & & \beta \searrow & & \gamma_0 \updownarrow & & \searrow & & \cdots \\
& & & R'_1 & & \overset{\alpha'}{\to} & & R'_2 & & \overset{\alpha'}{\to} & & R'_3 & \to & \cdots
\end{array} \tag{3.27}$$

where $\alpha = \gamma_0 \circ \beta : R_l \to R_{l+1}$ and $\alpha' = \beta \circ \gamma_0 : R'_l \to R'_{l+1}$. There are two series of noncommutative Atiyah-Ward ansatz solutions and denoted by R_l or R'_l. In both solutions, in the commutative limit, $b = f$. The simplest ansatz R_0 and R'_1 lead to the so called the *Corrigan-Fairlie-'t Hooft-Wilczek* (CFtHW) ansatz [3, 80, 85, 86].

- Noncommutative Atiyah-Ward ansatz solutions R_l

  Noncommutative Atiyah-Ward ansatz solutions R_l are represented by the explicit form of elements b_l, e_l, f_l, g_l as quasideterminants of $(l+1)\times(l+1)$ matrices:

$$
b_l = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \boxed{\Delta_0} \end{vmatrix}^{-1}, \quad f_l = \begin{vmatrix} \boxed{\Delta_0} & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0 \end{vmatrix}^{-1},
$$

$$
e_l = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \boxed{\Delta_{-l}} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0 \end{vmatrix}^{-1}, \quad g_l = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \boxed{\Delta_l} & \Delta_{l-1} & \cdots & \Delta_0 \end{vmatrix}^{-1}.
$$

$$
J_l = \left|\begin{array}{c|c|ccc|c} \boxed{0} & -1 & 0 & \cdots & 0 & \boxed{0} \\ \hline 1 & \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} \\ \hline 0 & \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1} \\ \hline \boxed{0} & \Delta_l & \Delta_{l-1} & \cdots & \Delta_1 & \boxed{\Delta_0} \end{array}\right|, \quad J_l^{-1} = \left|\begin{array}{c|ccc|c|c} \boxed{\Delta_0} & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} & \boxed{0} \\ \hline \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1} & 0 \\ \hline \Delta_l & \Delta_{l-1} & \cdots & \Delta_1 & \Delta_0 & 1 \\ \hline \boxed{0} & 0 & \cdots & 0 & -1 & \boxed{\Delta_0} \end{array}\right|.
$$

In the Mason-Woodhouse gauge,

$$
h_l = \left|\begin{array}{c|c|ccc|c|c} \boxed{0} & 1 & 0 & \cdots & 0 & 0 & \boxed{0} \\ \hline 0 & \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} & 0 \\ \hline 0 & \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1} & 0 \\ \hline 0 & \Delta_l & \Delta_{l-1} & \cdots & \Delta_1 & \Delta_0 & 0 \\ \hline \boxed{0} & 0 & 0 & \cdots & 0 & 1 & \boxed{1} \end{array}\right|, \quad h_l^{-1} = \left|\begin{array}{c|ccc|c|c} \boxed{\Delta_0} & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} & \boxed{0} \\ \hline \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1} & 0 \\ \hline \Delta_l & \Delta_{l-1} & \cdots & \Delta_1 & \Delta_0 & 0 \\ \hline \boxed{0} & 0 & \cdots & 0 & -1 & \boxed{1} \end{array}\right|.
$$

$$
\tilde{h}_l = \left|\begin{array}{c|c|ccc|c|c} \boxed{1} & 1 & 0 & \cdots & 0 & 0 & \boxed{0} \\ \hline 0 & \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} & 0 \\ \hline 0 & \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1} & 0 \\ \hline 0 & \Delta_l & \Delta_{l-1} & \cdots & \Delta_1 & \Delta_0 & 0 \\ \hline \boxed{0} & 0 & 0 & \cdots & 0 & 1 & \boxed{0} \end{array}\right|, \quad \tilde{h}_l^{-1} = \left|\begin{array}{c|c|ccc|c} \boxed{1} & -1 & 0 & \cdots & 0 & \boxed{0} \\ \hline 0 & \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} \\ \hline 0 & \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1} \\ \hline \boxed{0} & \Delta_l & \Delta_{l-1} & \cdots & \Delta_1 & \boxed{\Delta_0} \end{array}\right|.
$$

- Noncommutative Atiyah-Ward ansatz solutions R'_l

  Noncommutative Atiyah-Ward ansatz solutions R'_l are represented by the explicit form of elements b'_l, e'_l, f'_l, g'_l as quasideterminants of $l \times l$ matrices:

$$
\begin{aligned}
b'_l &= \begin{vmatrix} \boxed{\Delta_0} & \Delta_{-1} & \cdots & \Delta_{1-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 \end{vmatrix}, \quad f'_l = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{l-1} & \Delta_{l-2} & \cdots & \boxed{\Delta_0} \end{vmatrix}, \\
e'_l &= \begin{vmatrix} \Delta_{-1} & \Delta_{-2} & \cdots & \boxed{\Delta_{-l}} \\ \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{l-2} & \Delta_{l-3} & \cdots & \Delta_{-1} \end{vmatrix}, \quad g'_l = \begin{vmatrix} \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} \\ \Delta_2 & \Delta_1 & \cdots & \Delta_{3-l} \\ \vdots & \vdots & \ddots & \vdots \\ \boxed{\Delta_l} & \Delta_{l-1} & \cdots & \Delta_1 \end{vmatrix}.
\end{aligned}
$$

$$
J'_l = \left|\begin{array}{c:ccc:c:c} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} & -1 \\ \hdashline \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1} & 0 \\ \hdashline \Delta_l & \Delta_{l-1} & \cdots & \Delta_1 & \boxed{\Delta_0} & \boxed{0} \\ \hdashline 1 & 0 & \cdots & 0 & \boxed{0} & \boxed{0} \end{array}\right|, \quad J'^{-1}_l = \left|\begin{array}{c:c:ccc:c} \boxed{0} & \boxed{0} & 0 & \cdots & 0 & 1 \\ \hdashline \boxed{0} & \boxed{\Delta_0} & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} \\ \hdashline 0 & \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1} \\ \hdashline -1 & \Delta_l & \Delta_{l-1} & \cdots & \Delta_1 & \Delta_0 \end{array}\right|.
$$

In the Mason-Woodhouse gauge,

$$
h'_l = \left|\begin{array}{c:ccc:c} \Delta_{l-1} & \Delta_{l-2} & \cdots & \boxed{\Delta_0} & \boxed{0} \\ \hdashline \Delta_{-1} & \Delta_{-2} & \cdots & \boxed{\Delta_{-l}} & \boxed{1} \\ \hdashline \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta_{l-2} & \Delta_{l-3} & \cdots & \Delta_{-1} & 0 \end{array}\right|, \quad h'^{-1}_l = \left|\begin{array}{c:ccc:c:c} \boxed{0} & 0 & \cdots & 0 & -1 & \boxed{0} \\ \hdashline 0 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1} & 0 \\ \hdashline 1 & \Delta_{l-1} & \cdots & \Delta_1 & \Delta_0 & 0 \\ \hdashline \boxed{0} & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} & \boxed{1} \end{array}\right|,
$$

$$
\tilde{h}'_l = \left|\begin{array}{c:ccc:c} 0 & \Delta_1 & \cdots & \Delta_{3-l} & \Delta_{2-l} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \Delta_{l-1} & \cdots & \Delta_1 & \Delta_0 \\ \hdashline \boxed{1} & \boxed{\Delta_l} & \cdots & \Delta_2 & \Delta_1 \\ \hdashline \boxed{0} & \boxed{\Delta_0} & \cdots & \Delta_{2-l} & \Delta_{1-l} \end{array}\right|, \quad \tilde{h}'^{-1}_l = \left|\begin{array}{c:ccc:c:c} \boxed{1} & \Delta_l & \cdots & \Delta_2 & \Delta_1 & \boxed{0} \\ \hdashline 0 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} & 0 \\ \hdashline 0 & \Delta_1 & \cdots & \Delta_{3-l} & \Delta_{2-l} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \Delta_{l-1} & \cdots & \Delta_1 & \Delta_0 & 1 \\ \hdashline \boxed{0} & 0 & \cdots & 0 & -1 & \boxed{0} \end{array}\right|,
$$

where

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & \boxed{a_{33}} & \boxed{a_{34}} \\ a_{41} & a_{42} & \boxed{a_{43}} & \boxed{a_{44}} \end{vmatrix} := \left[\begin{array}{cc} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & \boxed{a_{34}} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & \boxed{a_{43}} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & \boxed{a_{44}} \end{vmatrix} \end{array} \right].$$

Because J is gauge invariant, this shows that the present Bäcklund transformation is not just a gauge transformation but a non-trivial one. These compact representations are proved by using identities of quasideterminants. (See also Appendix A in [22].)

The explicit form of the solutions can be represented in terms of quasideterminants whose elements Δ_r ($r = -l+1, -l+2, \cdots, l-1$) satisfy

$$\frac{\partial \Delta_r}{\partial z} = \frac{\partial \Delta_{r+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_r}{\partial w} = \frac{\partial \Delta_{r+1}}{\partial \tilde{z}}, \quad -l+1 \leq r \leq l-2 \quad (l \geq 2), \tag{3.28}$$

which implies that every element Δ_r is a solution of the noncommutative linear equation $(\partial_z \partial_{\tilde{z}} - \partial_w \partial_{\tilde{w}})\Delta_r = 0$.

The proof of these results can be made directly by using identities of quasideterminants only, such as, noncommutative Jacobi identity, homological relations, and Gilson-Nimmo's derivative formula [22, 23]. This implies that *noncommutative Bäcklund transformations are identities of quasideterminants.*

4 Noncommutative Ward's Conjecture

Here we briefly discuss reductions of noncommutative anti-self-dual Yang-Mills equation into lower-dimensional noncommutative integrable equations such as the noncommutative KdV equation. let us summarize the strategy for reductions of noncommutative anti-self-dual Yang-Mills equation into lower-dimensions. Reductions are classified by some ingredients such as a choice of gauge group a choice of symmetry a choice of gauge fixing and so on. Gauge groups are in general $GL(N)$. We have to take $U(1)$ part into account in noncommutative case. A choice of symmetry reduces noncommutative anti-self-dual Yang-Mills equations to simple forms. We note that noncommutativity must be eliminated in the reduced directions because of compatibility with the symmetry. Hence within the reduced directions, discussion about the symmetry is the same as commutative one. A choice of gauge fixing is the most important ingredient in this article which is shown explicitly at each subsection. The residual gauge symmetry sometimes shows equivalence of a few reductions. Constants of integrations in the process of reductions sometimes lead to parameter families of noncommutative reduced equations, however, in this article, we set all integral constants zero for simplicity.

Here, we present non-trivial reductions of noncommutative anti-self-dual Yang-Mills equation with $G = GL(2)$ to the noncommutative KdV equation via a $(2+1)$-dimensional integrable equation.

Let us start with the standard anti-self-dual Yang-Mills equation (2.13) with $G = GL(2,\mathbb{C})$) and impose the following translational invariance:

$$Y = \partial_{\tilde{z}}, \tag{4.1}$$

and impose the following non-trivial reduction conditions for the gauge fields:

$$\begin{aligned}
&A_{\tilde{w}} = O,\ A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ A_w = \begin{pmatrix} q & -1 \\ q_w + q \star q & -q \end{pmatrix}, \\
&A_z = \begin{pmatrix} (1/2)q_{w\tilde{w}} + q_{\tilde{w}} \star q + \alpha & -q_{\tilde{w}} \\ \phi & -(1/2)q_{w\tilde{w}} - q \star q_{\tilde{w}} + \alpha \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= \partial_w^{-1}[q_w, q_{\tilde{w}}]_\star, \qquad \partial_w^{-1} f(w) := \int^w dw' f(w'),\ \ \{A,B\}_\star := A \star B + B \star A, \\
\phi &= -q_z + \frac{1}{2}q_{ww\tilde{w}} + \frac{1}{2}\{q, q_{w\tilde{w}}\}_\star + \frac{1}{2}\{q_w, q_{\tilde{w}}\}_\star + q \star q_{\tilde{w}} \star q + [q, \partial_w^{-1}[q_w, q_{\tilde{w}}]_\star]_\star.
\end{aligned}$$

Then we get a noncommutative toroidal KdV equation [36, 81] by identifying $2q_w = u$:

$$u_z = \frac{1}{4}u_{ww\tilde{w}} + \frac{1}{2}\{u, u_{\tilde{w}}\}_\star + \frac{1}{4}\left\{u_{\tilde{w}}, \partial_w^{-1}u_{\tilde{w}}\right\}_\star + \frac{1}{4}\partial_w^{-1}[u, \partial_w^{-1}[u, \partial_w^{-1}u_{\tilde{w}}]_\star]_\star. \tag{4.2}$$

This equation has hierarchy and N-soliton solutions in terms of quasideterminant of Wronskian [34]. We note that under the ultrahyperbolic signature $(++--)$, all remaining coordinates among $z, w, \tilde{w}$ can be real [58].

Here if we take further reduction $\partial_w = \partial_{\tilde{w}}$, that is, dimensional reduction into $X = \partial_w - \partial_{\tilde{w}}$ direction, then the reduced equation coincides with the noncommutative KdV equation:

$$\dot{u} = \frac{1}{4}u''' + \frac{3}{4}\left(u' \star u + u \star u'\right). \tag{4.3}$$

where $(t, x) \equiv (z, w + \tilde{w})$ and $\dot{f} := \partial f/\partial t,\ f' := \partial f/\partial x$. We note that the gauge group is not $SL(2)$ but $GL(2)$ because A_z is not traceless. This implies $U(1)$ part of the gauge group plays a crucial role in the reduction also.

This noncommutative KdV equation has been studied by several authors and proved to possess infinite conserved quantities [8] in terms of Strachan's products [74] and exact multi-soliton solutions in terms of quasideterminants also [13, 34]. (See also [66, 68].)

5 Conclusion and Discussion

In this article, we have presented Bäcklund transformations for the noncommutative anti-self-dual Yang-Mills equation with $G = GL(2)$ and constructed from a simple seed solution a series of exact noncommutative Atiyah-Ward ansatz solutions expressed explicitly in

terms of quasideterminants. We have also discussed the origin of this transformation in the framework of noncommutative twistor theory.

These results could be applied also to lower-dimensional systems via the results on the noncommutative Ward's conjecture including noncommutative monopoles, noncommutative KdV equations and so on, and might shed light on a profound connection between higher-dimensional integrable systems related to twistor theory and lower-dimensional ones related to Sato's theory.

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A Brief Review of Quasideterminants

In this section, we make a brief introduction of quasi-determinants introduced by Gelfand and Retakh in 1992 [18] and present a few properties of them which play important roles in section 4. A good survey is e.g. [16] and relation between quasi-determinants and noncommutative symmetric functions is summarized in e.g. [17]. (See also, [46, 75])

Quasi-determinants are not just a noncommutative generalization of usual commutative determinants but rather related to inverse matrices.

Let $A = (a_{ij})$ be a $n \times n$ matrix and $B = (b_{ij})$ be the inverse matrix of A. Here all matrix elements are supposed to belong to a (noncommutative) ring with an associative product. This general noncommutative situation includes the Moyal or noncommutative deformation which we discuss in the main sections.

Quasi-determinants of A are defined formally as the inverse of the elements of $B = A^{-1}$:

$$|A|_{ij} := b_{ji}^{-1}. \tag{A.1}$$

In the commutative limit, this is reduced to

$$|A|_{ij} \longrightarrow (-1)^{i+j} \frac{\det A}{\det \tilde{A}^{ij}}, \tag{A.2}$$

where $\tilde{A}^{ij}$ is the matrix obtained from A deleting the i-th row and the j-th column.

We can write down more explicit form of quasi-determinants. In order to see it, let us recall the following formula for a square matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}, \tag{A.3}$$

where A and D are square matrices, and all inverses are supposed to exist. We note that any matrix can be decomposed as a 2×2 matrix by block decomposition where the diagonal parts are square matrices, and the above formula can be applied to the decomposed 2×2 matrix. So the explicit forms of quasi-determinants are given iteratively by the following formula:

$$\begin{aligned} |A|_{ij} &= a_{ij} - \sum_{i'(\neq i), j'(\neq j)} a_{ii'} ((\tilde{A}^{ij})^{-1})_{i'j'} a_{j'j} \\ &= a_{ij} - \sum_{i'(\neq i), j'(\neq j)} a_{ii'} (|\tilde{A}^{ij}|_{j'i'})^{-1} a_{j'j}. \end{aligned} \tag{A.4}$$

It is sometimes convenient to represent the quasi-determinant as follows:

$$|A|_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & & \boxed{a_{ij}} & & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}. \tag{A.5}$$

Examples of quasi-determinants are, for a 1×1 matrix $A = a$

$$|A| = a, \tag{A.6}$$

and for a 2×2 matrix $A = (a_{ij})$

$$\begin{aligned} |A|_{11} &= \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12}a_{22}^{-1}a_{21}, \quad |A|_{12} = \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} = a_{12} - a_{11}a_{21}^{-1}a_{22}, \\ |A|_{21} &= \begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix} = a_{21} - a_{22}a_{12}^{-1}a_{11}, \quad |A|_{22} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix} = a_{22} - a_{21}a_{11}^{-1}a_{12}, \end{aligned} \tag{A.7}$$

and for a 3×3 matrix $A = (a_{ij})$

$$\begin{aligned} |A|_{11} &= \begin{vmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} - (a_{12}, a_{13}) \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix} \\ &= a_{11} - a_{12} \begin{vmatrix} \boxed{a_{22}} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1} a_{21} - a_{12} \begin{vmatrix} a_{22} & a_{23} \\ \boxed{a_{32}} & a_{33} \end{vmatrix}^{-1} a_{31} \\ &\quad - a_{13} \begin{vmatrix} a_{22} & \boxed{a_{23}} \\ a_{32} & a_{33} \end{vmatrix}^{-1} a_{21} - a_{13} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} a_{31}, \end{aligned} \tag{A.8}$$

and so on.

Quasideterminants have various interesting properties similar to those of determinants. Among them, the following ones play important roles in this article. In the block matrices given in these results, lower case letters denote single entries and upper case letters denote matrices of compatible dimensions so that the overall matrix is square. (By using boxes, it becomes easier to calculate various identities. Such calculations are fully presented in e.g. [21, 62].)

- Noncommutative Jacobi identity [18, 21]

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}.$$

- Homological relations [18]

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & \boxed{h} & i \end{vmatrix} = \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} \begin{vmatrix} A & B & C \\ D & f & g \\ 0 & \boxed{0} & 1 \end{vmatrix}, \quad \begin{vmatrix} A & B & C \\ D & f & \boxed{g} \\ E & h & i \end{vmatrix} = \begin{vmatrix} A & B & 0 \\ D & f & \boxed{0} \\ E & h & 1 \end{vmatrix} \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix}$$

- *Gilson-Nimmo's derivative formula* [21]

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = \begin{vmatrix} A & B' \\ C & \boxed{d'} \end{vmatrix} + \sum_{k=1}^{n} \begin{vmatrix} A & (A_k)' \\ C & \boxed{(C_k)'} \end{vmatrix} \begin{vmatrix} A & B \\ e_k^t & \boxed{0} \end{vmatrix},$$

where A_k is the kth column of a matrix A and e_k is the column n-vector (δ_{ik}) (i.e. 1 in the kth row and 0 elsewhere).

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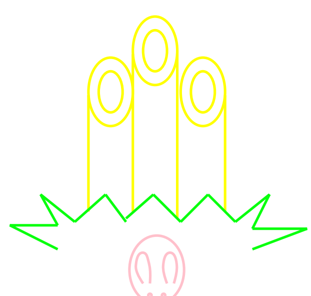

Figure 1: Kadomatsu with a rabbit